FINAL EXAM ISI BANGALORE NOVEMBER 2015

M.MATH LINEAR ALGEBRA 100 POINTS

Notes.

(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.

(b) Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers $\mathbb{C}$ = complex numbers.

(d) All vector spaces are assumed to be finite dimensional, unless mentioned otherwise.

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1. [12 points] Let $T\colon V \to V$ be a linear map of vector spaces and let $W \subset V$ be a T-invariant subspace.
   (i) Describe how T induces a natural linear map $\overline{T}\colon V/W \to V/W$.
   (ii) Prove or disprove: If $T\big|_W$ is diagonalizable and $\overline{T}$ is diagonalizable then T is diagonalizable.

2. [12 points] Let F be a field, $X = (x_{ij})$ an $m \times n$ matrix over F, and let $a_1, \dots, a_m, b_1, \dots, b_n$ be elements of F. Let $Y = (y_{ij})$ be the $m \times n$ matrix given by $y_{ij} = x_{ij} + a_i + b_j$. Prove that $|\mathrm{rank}(X) - \mathrm{rank}(Y)| \leq 2$.

3. [12 points] Let $V_1 \xrightarrow{T} V_2 \xrightarrow{S} V_3$ be linear maps of vector spaces.
   (i) Define what it means for the above sequence of maps to be exact.
   (ii) If $ST = 0$, prove that $\mathrm{rank}(S) + \mathrm{rank}(T) \leq \dim(V_2)$.

4. [12 points]
   (i) Given unit vectors $u_1, \dots, u_m$ in $\mathbb{R}^n$, show that there exists a pair u_i, u_j (with $i \neq j$) such that $u_i \cdot u_j \geq \frac{-1}{m-1}$.
   (ii) Let $v_1, v_2, v_3 \in \mathbb{R}^n$ be nonzero vectors and let θ_{ij} be the angle between v_i and v_j. Show that $\min(\theta_{12}, \theta_{23}, \theta_{31}) \leq 2\pi/3$ with equality iff v_1, v_2, v_3 are coplanar with $\theta_{12} = \theta_{23} = \theta_{31}$.

5. [28 points] TRUE or FALSE: In each of the following statements, decide whether it is true or false and give brief explanations for your answer. You will get credit only if your explanation is correct.
   (i) If A is a symmetric invertible matrix with coefficients in $\mathbb{C}$, then $A = Q^tQ$ for some invertible matrix Q over $\mathbb{C}$.
   (ii) If A is a Hermitian symmetric invertible matrix, then $A = Q^*Q$ for some invertible matrix Q over $\mathbb{C}$.
   (iii) If A, B are real symmetric matrices having the same characteristic polynomial, then A is similar to B over $\mathbb{R}$.
   (iv) If $\{v_i\}_{i=1}^n$ is an arbitrary basis of a Hermitian space $(V, \langle\ ,\ \rangle)$ and $T\colon V \to V$ is an isomorphism such that $\langle v_i, v_j\rangle = \langle Tv_i, Tv_j\rangle$ then T is unitary.

6. [12 points] Classify upto similarity, all 5×5 matrices over $\mathbb{C}$ whose minimal polynomial is given by $p(t) = (t+1)(t-1)^2$.

7. [12 points] Let F be a field and let $a_0, \ldots, a_{n-1} \in F$. Give an example of an $n \times n$ matrix A over F such that $A^n + a_{n-1}A^{n-1} + \cdots + a_0$ equals the zero matrix.